ON THE STRING INTERPRETATION OF THE $t\bar{t}$-GEOMETRY

César Gómez and Esperanza López

Instituto de Matemáticas y Física Fundamental,
Serrano 123, 28006 Madrid (Spain)

Abstract

We derive the $t\bar{t}$-equations for generic $N = 2$ topological field theories as consistency conditions for the contact term algebra of topological strings. A generalization of the holomorphic anomaly equation, known for the critical $\hat{c} = 3$ case, to arbitrary non critical topological strings is presented. The interplay between the non-trivial cohomology of the b-antighost, gravitational descendants and $\bar{t}$-dependence is discussed. The physical picture emerging from this study is that the $\bar{t}$ (background) dependence of topological strings with non-trivial cohomology for the b-antighost, is determined by gravitational descendants.

1 Introduction

An important issue in string dynamics is certainly the study of the geometry of the space of two dimensional quantum field theories, a question which is intimately connected with the main problem of string background independence [1]. In the simplest setting of topological string theory [2] a partial understanding of the background independence problem [3] can be reach through the study of the holomorphic anomaly [4, 5]. This is in part due to the more precise knowledge, we have in this case, of the geometry of the "theory space", which turns out to be a sort of generalized $N = 2$ special geometry [6] known as topological-antitopological, $t\bar{t}$, fusion [7].

The general structure of this geometry is defined by a vector bundle with the base space parametrizing the different couplings, and fiber V, the BRST cohomology of the corresponding topological field theory. The main ingredient in the characterization of the $t\bar{t}$-geometry is the existence in V of a hermitean scalar product $\langle \, , \, \rangle$ such that

$$\{Q, Q^*\} = H \tag{1}$$

for Q^* the adjoint of the BRST charge Q, and H the hamiltonian. Equation (1) together with the nilpotency $Q^2 = Q^{*2} = 0$ defines the SUSY $N = 2$ algebra or, in more mathematical terms, a Hodge system. Denoting by $|i\rangle$ a topological basis, i.e. the cohomology of Q, the $t\bar{t}$-metric is defined by

$$\langle \bar{j} | i \rangle = g_{i\bar{j}} \tag{2}$$

The derivation of the $t\bar{t}$-geometry requires now to introduce a connection by

$$\langle \bar{k} | D_i | j \rangle = 0 \,, \qquad \langle \bar{k} | \bar{D}_{\bar{i}} | j \rangle = 0 \tag{3}$$

with respect to which the metric $g_{i\bar{j}}$ is covariantly constant. The $t\bar{t}$-equations for this connection are

$$[D_i, \bar{D}_{\bar{j}}] = -[C_i, \bar{C}_{\bar{j}}] \tag{4}$$

with the $C's$ the ring structure constants.

1.1 Special geometry. In order to see the strong analogy with special geometry let us consider the example of a Calabi Yau 3-fold M where the space of couplings is identified with the moduli of complex structures, X. In this case the relevant BRST cohomology states correspond to elements in $H^{2,1}(M)$. The role of the vacuum, making possible the map from operators to states, is played by the holomorphic top form $\Omega^{(3,0)}$ of M, with the state-operator map being determined by the isomorphism between $H^{0,1}(TM)$ and $H^{2,1}(M)$. The full BRST cohomology is given by $H^3(M) = H^{(3,0)} \otimes H^{(2,1)} \otimes H^{(1,2)} \otimes H^{(0,3)}$, whose elements will be denoted respectively as Ω, V_l, $V_{\bar{l}}$, $\bar{\Omega}$. The hermitean scalar product of forms α, $\beta \in H^3$ is defined by means of the simplectic form on M

$$\langle \alpha, \beta \rangle = \int \alpha \wedge \beta \tag{5}$$

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and therefore the adjoint of an state (p,q) is an element $(3-p, 3-q)$.

Infinitesimal motion on the moduli space of complex structures mix (p,q)-forms with $(p\pm 1, q\mp 1)$-forms. In particular, the top form Ω mixes only with $(2,1)$-forms

$$\partial_i \Omega = V_i + a_i \Omega \tag{6}$$

with a_i certain function, not globally holomorphic.

Using now the definition (3) of covariant derivative and the inner product (5), we deduce that the projection of $\partial_i V_j$ on $H^{(2,1)}$-forms defines the connection on X

$$\partial_i V_j = A_{ij}^k + ... \tag{7}$$

where by points we mean degree $(1,2)$ and $(3,0)$ contributions. Special geometry allow to express the Yukawa couplings in the following way

$$C_{ijk} = -\int \Omega \wedge \partial_i \partial_j \partial_k \Omega = \int \partial_i \Omega \wedge \partial_j \partial_k \Omega \tag{8}$$

From this we easily get

$$\partial_j(\partial_k \Omega) = C_{jk}^{\bar{l}} V_{\bar{l}} + (2,1) + (3,0) \tag{9}$$

and therefore

$$\partial_i V_j = C_{ij}^{\bar{l}} V_{\bar{l}} + A_{ij}^k V_k + (3,0) \tag{10}$$

Under an infinitesimal motion in the $\bar{t}$-direction on X, and using (6)-(9), equation (10) give raise to the curvature equations for the connection A_{ij}^k

$$\partial_{\bar{l}} A_{ij}^k = G_{j\bar{l}} \delta_i^k - \bar{C}_{\bar{l}}^{kn} C_{ijn} \tag{11}$$

where $G_{j\bar{l}} = \partial_{\bar{l}} a_j$ (see eq.(6)). Equation (11) is a particular case of the general $t\bar{t}$-equation (4).

1.2 String representation. The string interpretation of the $t\bar{t}$-geometry is based on the following general philosophy. Given a topological field theory parametrized by the couplings $(t_i, \bar{t}_i)$ the variation of the correlators under small changes of the couplings is given by string amplitudes of the corresponding topological matter theory coupled to topological gravity. The reason for this is, of course, that a variation of the couplings corresponds to integrate over the world sheet the perturbing operator. This is clear from the lagrangian representation of a perturbed TFT

$$\mathcal{L} = \mathcal{L}_0 + \sum_i t_i \int \phi_i^{(2)} + \sum_{\bar{i}} \bar{t}_{\bar{i}} \int \bar{\phi}_{\bar{i}}^{(2)} \tag{12}$$

where

$$\phi_i^{(2)} = \{Q^*, [\bar{Q}^*, \phi_i]\} , \qquad \bar{\phi}_{\bar{i}}^{(2)} = \{Q, [\bar{Q}, \bar{\phi}_{\bar{i}}]\} \tag{13}$$

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and ϕ_i, $\bar{\phi}_{\bar{i}}$ are respectively chiral and antichiral primary fields. Therefore, the variation of correlators of the TFT under small changes of couplings implies the definition of a form on the moduli space of the punctured Riemann surface, and this is string theory. In this framework the $t\bar{t}$-connection on the space of couplings should be determined by means of contact terms, moreover the consistency conditions of the contact term algebra should correspond to geometrical constraints. In topological string theories [8] these contact terms are computed by the cancel propagator argument and they correspond to the contribution at the boundary of the moduli space defined when two punctures collide. This kind of computation strongly depends on the way the string measures on the moduli space have been defined, and in particular on the type of (b, c) ghost system we use. In the case we are interested in associating forms on the moduli of punctured Riemann surfaces to infinitesimal changes of the couplings $(t_i, \bar{t}_i)$, we are forced to use as the b antighost the supercharge Q^* which is the adjoint of the BRST charge. The so defined string differs from the standard bosonic string in a crucial aspect, namely the cohomology of b is now non trivial [5]. We will call Hodge strings those for which the b antighost possess non trivial cohomology and such that the pair (Q, b) satisfies the usual Hodge relations[1].

Summarizing, the string interpretation of the $t\bar{t}$-geometry that we want to present in this paper will be based on the following dictionary [9]

$$t\bar{t} - connection \iff Contact \quad Terms$$
$$t\bar{t} - equations \iff Consistency \quad Conditions$$

1.3 Hodge equivariance and background independence. The simplest consequence of the non trivial cohomology for the b antighost is the absence, in the equivariant cohomology defined by the Hodge pair (Q, b), of gravitational descendants. To define the physical states in this equivariant cohomology is more than we need for constructing good string amplitudes independent of the local world sheet coordinates. Moreover these Hodge strings present a severe form of BRST anomaly, namely the holomorphic anomaly [4, 5], which implies a $\bar{t}$-dependence of the amplitudes. Taking into account that both, the appearance of the holomorphic anomaly and the absence in "Hodge" equivariance of gravitational descendants share a common origin, namely the non trivial cohomology of the b antighost, it is natural to try to connect them. A possible way to do it is trying to match the $\bar{t}$-dependence with the contribution of gravitational descendants. In our approach this phenomena shows up in the form of new mixed $t\bar{t}$-contact terms.

The plan of the paper is as follows. In section 2 we introduce a contact term algebra whose consistency conditions imply the whole set of $t\bar{t}$-equations. We will discuss also what is the meaning of Hodge strings. In section 3 we will propose a generalization of the holomorphic anomaly, presented in [5] for $\hat{c} = 3$ topological strings, to topological strings with arbitrary $\hat{c}$.

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[1]We thank A.Losev for stressing to us the connection with Hodge theory

2 Contact Terms and $(t, \bar{t})$-Fusion

In this section we will proceed to derive the $t\bar{t}$-geometry from a contact term algebra. We will work in the following general setting. Given a generic two dimensional topological field theory, we consider the $(t\bar{t})$ space of couplings defined by $N = 2$ preserving perturbations. Our aim will be to find the behaviour of the metric (2) under these perturbations in terms of the contact terms of a topological string.

2.1 $t\bar{t}$-connection and Contact Terms

The $t\bar{t}$-connection is defined in reference [7] by the condition

$$\langle \bar{j}|D_i|k\rangle = 0 \tag{14}$$

This corresponds to the standard Levi-Civita definition of connection, where the variation of the physical state $|k\rangle$, induced by the $N = 2$ preserving perturbation $\delta t_i \int \phi_i^{(2)}$, is orthogonally projected, with respect to the hermitean scalar product, on the basis of BRST physical states. By a contact term representation of (14), we mean

$$\langle \bar{j}|\partial_i|k\rangle = \langle \bar{j}|C(i, k)\rangle \tag{15}$$

with $C(i, k)$ a contact term defined in some topological string theory.

Before entering into the explicit definition of these contact terms, we will restrict them by imposing some consistency conditions. These conditions will be motivated by the string interpretation of these contact terms [8, 13, 14]. The string meaning of $|C(i, k)\rangle$ can be symbolically represented as follows

$$|C(i, k)\rangle = \int i \, |k\rangle \tag{16}$$

where we are thinking $|k\rangle$ as the state created by inserting at the origin of the disk the field ϕ_k, and $\int i$ as the integration of $\phi_i^{(2)}$ in a infinitesimal neighbourhood of the insertion point of ϕ_k. The consistency conditions are now determined by imposing independence of the order of integration

$$\int i \int j \, |k\rangle = \int j \int i \, |k\rangle \tag{17}$$

The contribution to each term of (17) is given by

$$\int i \int j|k\rangle = \int c(j, i)|k\rangle + \int i|C(j, k)\rangle \tag{18}$$

where we introduce an explicit difference between operator-operator contact terms $c(i, j)$ and the operator-state contact terms $|C(i, j)\rangle$ defined by (16).

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The relation between $c(i,j)$ and $|C(i,j)\rangle$ will reflects, of course, the standard state-operator relation or equivalently the way we define a reference vacuum state. In twisted topological field theories, the definition of the vacuum requires to soak up fermionic zero modes and therefore requires the insertion at the origin of the disk of some spectral flow like operators. Denoting generically by Φ the operator used in the definition of the vacuum, we can define the operator-operator contact term $c(i,j)$ by the following decomposition

$$|C(i,j)\rangle \equiv c(i,j)|\Phi\rangle + \phi_j|C(i,\Phi)\rangle \qquad (19)$$

where the second term in the r.h.s. of (19) represents the state obtained by inserting ϕ_j on the perturbed "vacuum".

At this point we should stress the difference between a marginal perturbation, which preserves charge conservation, and a massive perturbation. When the $U(1)$ charge of the $N=2$ algebra is conserved, the variation of the vacuum state is proportional to itself

$$|C(i,\Phi)\rangle = f_i|\Phi\rangle \qquad (20)$$

and in consequence the vacuum $|\Phi\rangle$ defines a line subbundle, $\mathcal{L}$, over the moduli space of marginal perturbations. In this case the $t\bar{t}$-geometry reduces to special geometry and, in particular, the existence of $\mathcal{L}$ translate into the existence of a Kähler potential from which to derive all the relevant geometrical quantities. The existence of $\mathcal{L}$ is also important for the definition of covariant string amplitudes, which will be discussed in the next section.

Under a generic massive perturbation breaking charge conservation, the variation of the vacuum can have projection into any harmonic state[2]

$$|C(i,\Phi)\rangle = A_{i0}^n|n\rangle \qquad (22)$$

and the notion of vacuum subbundle disappear.

In the context of Landau-Ginzburg theories [10], in which the chiral fields are given in terms of the superpotential W by

$$\phi_i = \frac{\partial W}{\partial t_i} \qquad (23)$$

it is safe to assume for the operator-operator contact terms, generically defined as $c(i,j) \equiv \frac{\partial \phi_j}{\partial t_i}$, the symmetry condition

$$c(i,j) = c(j,i) \qquad (24)$$

We will suppose that this holds in general, including all possible sources of asymmetric contributions to the contact terms in the last piece of equation (19). Notice that the analog

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[2]Let us note that

$$|C(i,P)\rangle = |C(i,\Phi)\rangle \qquad (21)$$

where P is the puncture operator. This is due to $c(i,P)=0$ in any TFT.

of this asymmetry in pure topological gravity [8] results from localizing the curvature at the insertion points, with Φ defining there the curvature insertion operator. In our formal definition of the contact term algebra, we transfer the whole problem of the asymmetry of contact terms into the general state-operator relation.

2.2 $t\bar{t}$-geometry and Contact Term Algebra

Our task now will be, assuming (15), to derive the $t\bar{t}$-geometry [7] using only consistency conditions of type (17). In order to do that, we need to introduce mixed topological-antitopological contact terms. Our philosophy in this section will be to introduce formally these contact terms and only, after solving the consistency conditions and matching the $t\bar{t}$-geometry, to look for a proper string representation of these mixed contact terms. We will use for operator-state contact terms notation (16). Taking into account that we are looking for the contact term algebra of a topological string theory we will work, from the beginning, with the whole tower of gravitational descendants, $\sigma_n(i)$ with $n > 0$ and i running over the chiral primary fields. The topological part of the contact term algebra is defined by

$$\int i \, |j\rangle = A_{ij}^k |k\rangle \tag{25}$$

$$\int \sigma_n(i) \, |\sigma_m(j)\rangle = A_{ij}^k |\sigma_{n+m}(k)\rangle + C_{ij}^l |\sigma_{n+m-1}(l)\rangle \ , \quad n+m > 0$$

with C_{ij}^l and A_{ij}^k some unknown tensors. From (25) and (15) we observe that the tensor A_{ij}^k will play the role of the $t\bar{t}$-connection. The second piece in (25.2) is the standard one we expect for topological matter coupled to topological gravity [11]. Now we complete (25) with the following mixed $t\bar{t}$-contact terms

$$\int \bar{a} |\sigma_n(i)\rangle = H_{\bar{a}i}^l |\sigma_{n+1}(l)\rangle \tag{26}$$

$$\int \sigma_n(i) |\bar{a}\rangle = \widetilde{H}_{i\bar{a}}^l |\sigma_{n+1}(l)\rangle$$

with the $\bar{a}$'s one to one related to the $\bar{t}$-$N=2$ preserving perturbations, and their associated states $|\bar{a}\rangle$ to be determined in the process of solving the consistency conditions[3]. The tensors $H_{\bar{a}i}^l$ and $\widetilde{H}_{i\bar{a}}^l$ are in principle different. The main feature of (26) is the appearance of gravitational descendants (consider the case $n = 0$) in the topological-antitopological fusion. In fact, a natural way to read equation (26), that we will discuss latter, is as a procedure to associate with pure matter states their gravitational descendants.

One more ingredient is still necessary before entering to solve the consistency conditions (17). We will generalize the contact terms (25) and (26) to the case in which

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[3]The states $|\bar{a}\rangle$ should not be confused with antitopological Ramond vacua.

functions $f(t,\bar{t})$ are present, in the following way

$$
\begin{aligned}
\int i\ (f(t,\bar{t})|A\rangle) &= \partial_i f(t,\bar{t})|A\rangle + f(t,\bar{t})|C(i,A)\rangle \\
\int \bar{a}\ (f(t,\bar{t})|A\rangle) &= \partial_{\bar{a}} f(t,\bar{t})|A\rangle + f(t,\bar{t})|C(\bar{a},A)\rangle \\
\int \sigma_n(i)\ (f(t,\bar{t})|A\rangle) &= \partial_i f(t,\bar{t})|\sigma_n(A)\rangle + f(t,\bar{t})|C(\sigma_n(i),A)\rangle
\end{aligned}
\tag{27}
$$

with A a generic state. Notice that in general the tensors appearing in (25) and (26) will depend on the coordinates $(t,\bar{t})$ of the space of theories. The logic for for these rules is the equivalence between the insertion of a field and the derivation with respect to the corresponding t or $\bar{t}$ parameter. Considering that we want to study the $t\bar{t}$ space and not the full phase space available for the topological string, the derivation rule associated to arbitrary gravitational descendants (27.3) should only involve their t-part.

2.3   Consistency Conditions: Computations

We pass now to study systematically the consistency conditions for the contact term algebra defined by equations (25) and (26). From the symmetry of the operator-operator contact terms (24), the consistency conditions

$$
\int A \int B\ |C\rangle = \int B \int A\ |C\rangle
\tag{28}
$$

for A, B, C arbitrary operators, reduce to

$$
\int A \left(\int B\ |C\rangle \right) = \int B \left(\int A\ |C\rangle \right)
\tag{29}
$$

We will use from now on this simplified form.

Let us begin considering

$$
\int \sigma_1(i) \left(\int \sigma_1(j)\ |k\rangle \right) = \int \sigma_1(j) \left(\int \sigma_1(i)\ |k\rangle \right)
\tag{30}
$$

From the contact term algebra (25)-(26) and the derivation rules (27), we get

$$
\int \sigma_1(i)(\int \sigma_1(j)\ |k\rangle) = \partial_i A^l_{jk}|\sigma_2(l)\rangle + A^n_{jk}(A^l_{in}|\sigma_2(l)\rangle + C^l_{in}|\sigma_1(l)\rangle)+
$$
$$
+\partial_i C^l_{jk}|\sigma_1(l)\rangle + C^n_{jk}(A^l_{in}|\sigma_1(l)\rangle + C^l_{in}|l\rangle) = i \leftrightarrow j
\tag{31}
$$

Defining, according to (14) and (15), a covariant derivative by

$$
D_i \equiv \partial_i - A_i
\tag{32}
$$

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we obtain from (31) both the flatness condition

$$[D_i, D_j] = 0 \tag{33}$$

and the integrability condition for the tensor C_{ij}^k

$$D_i C_{jk}^l = D_j C_{ik}^l \tag{34}$$

where the connection A acts only in the state indices k, l. The connection associated to the operator indices i, j should be given by $c(i,j)$ which, being symmetric, cancels from expression (34). It also follows from the consistency condition (31) the associativity of the tensor C_{ij}^k

$$C_{ik}^n C_{jn}^l = C_{jk}^n C_{in}^l \tag{35}$$

Equations (34) and (35) imply that C_{ij}^k are the structure constants of the TFT. These equations, together with (33), are the t-part of the $t\bar{t}$-equations [7].

We study next a consistency condition involving $t\bar{t}$-contact terms

$$\int \bar{a}(\int \sigma_1(i) \, |j\rangle) = \int \sigma_1(i)(\int \bar{a} \, |j\rangle) \tag{36}$$

From

$$\int \bar{a}(\int \sigma_1(i) \, |j\rangle) = \partial_{\bar{a}} A_{ij}^k |\sigma_1(k)\rangle + H_{\bar{a}k}^l (A_{ij}^k |\sigma_2(l)\rangle + C_{ij}^k |\sigma_1(l)\rangle) + \partial_{\bar{a}} C_{ij}^k |k\rangle$$

$$\int \sigma_1(i)(\int \bar{a} \, |j\rangle) = \partial_i H_{\bar{a}j}^k |\sigma_2(k)\rangle + H_{\bar{a}j}^n (A_{in}^l |\sigma_2(l)\rangle + C_{in}^l |\sigma_1(l)\rangle) \tag{37}$$

we get a $t\bar{t}$-type equation for the connection A_i

$$\partial_{\bar{a}} A_{ij}^k = [H_{\bar{a}}, C_i]_j^k \tag{38}$$

the following constrain for the tensor $H_{\bar{a}}$

$$\partial_i H_{\bar{a}j}^k + A_{in}^k H_{\bar{a}j}^n - A_{ij}^n H_{\bar{a}n}^k \equiv D_i H_{\bar{a}j}^k = 0 \tag{39}$$

and the holomorphicity of C_{ij}^k

$$\partial_{\bar{a}} C_{ij}^k = 0 \tag{40}$$

which is satisfied if C_{ij}^k are the topological structure constants.

The requirement

$$\int \bar{a}(\int \bar{b} \, |i\rangle) = \int \bar{b}(\int \bar{a} \, |i\rangle) \tag{41}$$

implies more constrains on the tensor $H_{\bar{a}}$, namely

$$\partial_{\bar{a}} H_{\bar{b}i}^k = \partial_{\bar{b}} H_{\bar{a}i}^k \tag{42}$$
$$H_{\bar{a}i}^n H_{\bar{b}n}^k = H_{\bar{b}i}^n H_{\bar{a}n}^k$$

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Before using (39) and (42) to solve $H_{\bar{a}}$, we need another piece of information coming from the contact term algebra. In order to have a coherent interpretation of the contact term

$$\int P|\bar{a}\rangle = \widetilde{H}^l_{0\bar{a}}|\sigma_1(l)\rangle \tag{43}$$

for P the puncture operator, the simplest choice is to identify the "formal" states $|\bar{a}\rangle$ with

$$|\bar{a}\rangle = M^l_{\bar{a}}|\sigma_2(l)\rangle \tag{44}$$

From this we immediately get

$$\int i|\bar{a}\rangle = \int i(M^l_{\bar{a}}|\sigma_2(l)\rangle) = (\partial_i M^l_{\bar{a}} + M^n_{\bar{a}} A^l_{in})|\sigma_2(l)\rangle + C^l_{in} M^n_{\bar{a}}|\sigma_1(l)\rangle \tag{45}$$

The consistency of (45) with the contact term

$$\int i|\bar{a}\rangle = \widetilde{H}^l_{i\bar{a}}|\sigma_1(l)\rangle \tag{46}$$

requires

$$D_i M^l_{\bar{a}} = 0 \tag{47}$$
$$\widetilde{H}^l_{i\bar{a}} = C^l_{in} M^n_{\bar{a}}$$

Assuming now that the matrix M is invertible, we obtain from equation (47.1) that the connection A_i is given by

$$A^k_{ij} = (\partial_i M^{\bar{a}}_j) M^k_{\bar{a}} \tag{48}$$

where $M^{\bar{a}}_j M^k_{\bar{a}} = \delta^k_j$. The matrix $M^l_{\bar{a}}$ can be understood as providing an isomorphism between the topological and antitopological sectors. Therefore, from eq. (44), we can now interpret the formal states $|\bar{a}\rangle$, introduced in (26.2), as a gravitationally dressed version of the antitopological basis.

Using $M^l_{\bar{a}}$ as a topological-antitopological change of basis, it is convenient to redefine $H_{\bar{a}}$ in terms of a new tensor $\bar{C}^{\bar{c}}_{\bar{a}\bar{b}}$ as follows

$$H^l_{\bar{a}i} = \bar{C}^{\bar{c}}_{\bar{a}\bar{b}} M^{\bar{b}}_i M^l_{\bar{c}} \tag{49}$$

From (39) and (42), the tensor $\bar{C}^{\bar{c}}_{\bar{a}\bar{b}}$ should satisfy

$$\begin{aligned}
\bar{C}^{\bar{n}}_{\bar{a}\bar{c}} \bar{C}^{\bar{d}}_{\bar{b}\bar{n}} &= \bar{C}^{\bar{n}}_{\bar{b}\bar{c}} \bar{C}^{\bar{d}}_{\bar{a}\bar{n}} \\
D_{\bar{a}} \bar{C}^{\bar{d}}_{\bar{b}\bar{c}} &= D_{\bar{b}} \bar{C}^{\bar{d}}_{\bar{a}\bar{c}} \\
\partial_i \bar{C}^{\bar{c}}_{\bar{a}\bar{b}} &= 0
\end{aligned} \tag{50}$$

which are the defining relations for the structure constants of an antitopological theory. Let us remark that (50) does not oblige the $\bar{C}_{\bar{a}}$ to be the structure constants of the conjugate antitopological theory, i.e. $\bar{C}_{\bar{a}} = (C_a)^*$. The only condition on the antitopological

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theory is that its ring of observables has the same dimension as the one of the topological theory with which it mixes.

Once we have fixed all the tensors appearing in the contact term algebra, we can derive from (38) and (49), the $t\bar{t}$-equation for the connection

$$\partial_{\bar{a}} A_{ij}^k = [\bar{C}_{\bar{a}}, C_i]_j^k \tag{51}$$

Thus we have proved that the complete set of $t\bar{t}$-equations appear as the solution of the consistency conditions of the contact term algebra (25), (26).

Before finishing this section, we would like to make some comments. A very important ingredient in solving the consistency conditions for the algebra (25) and (26), was to suppose the symmetry of the operator-operator contact terms (24). Indeed, this is consistent with the solution we have obtained. Let us consider the contact term between primary fields, according to (19)

$$c(i,j) \equiv A_{ij}^k \phi_k - A_{i0}^n C_{jn}^k \phi_k = g_{0\bar{b}} (\partial_i C_{j\bar{a}}^{\bar{b}}) g^{\bar{a}k} \phi_k \tag{52}$$

which is clearly symmetric in i, j (see eq. (34)). This extends trivially to the contact terms involving gravitational descendants because the structure constants satisfy

$$C_{ij}^k = C_{ji}^k \tag{53}$$

Based on these facts, we have assumed that $c(\bar{a}, i) = c(i, \bar{a})$[4].

All the consistency conditions are immediately satisfied by the solution presented, with two exceptions

$$\int i \int \sigma_n(j) |k\rangle = \int \sigma_n(j) \int i |k\rangle , \quad n > 0 \tag{54}$$

$$\int i \int \bar{a} |j\rangle = \int \bar{a} \int i |j\rangle$$

which both will involve factorization terms. The necessity for including factorization terms at the level of consistency conditions is already present in the simplest case of contact term algebra, namely in pure topological gravity. In that case, and due to the asymmetry of the contact algebra, it is not possible to fulfill $\int \hat{P} \int \hat{\sigma}_n |\hat{P}\rangle = \int \hat{\sigma}_n \int \hat{P} |\hat{P}\rangle$[5] without taking into account factorization terms.

A general contribution from a factorization term is

$$\int A \int B |C\rangle = \mathcal{B}_B^{\alpha\beta} C_{\beta AC} |\alpha\rangle + \dots \tag{55}$$

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[4]Notice that the operator involved in these operator-operator contact terms is the one which should define the state $|\bar{a}\rangle$ given by equation (44). In fact the state-operator relation for the antitopological sector involved in the contact term algebra should formally be defined by (44) and the condition $c(\bar{a}, i) = c(i, \bar{a})$.

[5]The asymmetry is originated by the curvature factor that defines the corrected operators $\hat{\sigma}_n = e^{\frac{2}{3}(n-1)\pi} \sigma_n$, where π is the conjugate of the Liouville field.

where $\mathcal{B}_B^{\alpha\beta}$ is the factorization tensor, the indices α, β run in principle over gravitational descendants, and $C_{\beta AC}$ are 3-point functions. The dots in (55) mean the usual contact term contributions. By Witten's recursion relations [2], $C_{\beta AC}$ vanishes unless the three indices are primary (see next section). Therefore the only consistency conditions in which factorization terms can appear are (54), as it is necessary.

The solution of (54.1) involves a factorization tensor satisfying

$$\mathcal{B}_{(n,j)}^{(n-2,l)r} C_{rik} \equiv \partial_j C_{ik}^l + A_{jr}^l C_{ik}^r \ , \quad n > 0 \tag{56}$$

More interesting are the factorization terms contributing to (54.2), which are given by

$$\mathcal{B}_{\bar{a}}^{ij} = -H_{\bar{a}r}^i \eta^{rj} = -\bar{C}_{\bar{a}}^{ij} \tag{57}$$

where η_{ij} is the topological metric. The consistency conditions applied to a general string amplitude impose additional restrictions on the factorization tensors, which in the case of (57) imply that the metric η_{ij} is covariantly constant

$$D_k \eta_{ij} = 0 \tag{58}$$

The hermitean metric $g_{i\bar{j}}$ defined in (2) can be represented in the following way

$$g_{i\bar{j}} = \eta_{ik} M_{\bar{j}}^k \tag{59}$$

in consequence, from (47) and (58), it is also covariantly constant. The existence on the space of theories of a topological and a hermitean compatible metrics, i.e. both of them covariantly constants, was showed in [12] to be equivalent to the $t\bar{t}$-geometry.

The appearance of factorization terms for the $\bar{a}$ operators can be motivated by the $n=2$ gravitational index that they, heuristically, seems to carry as reflected in $|\bar{a}\rangle = M_{\bar{a}}^l |\sigma_2(l)\rangle$ (see eq.(44)). The main task of section 3 will be to explore the consequences of the factorization term (57) for the $\bar{a}$ operator. We will propose there a generalization of the holomorphic anomaly equation, discovered in [5] for $\hat{c}=3$ topological strings, to a generic topological string theory.

2.4 Hodge Strings and Gravitational Descendants

One of the main ingredients in the definition of string amplitudes is the couple (Q, b) with Q the BRST operator and b the antighost. Their algebraic relations are given by:

$$Q^2 = b_0^2 = 0 \tag{60}$$
$$\{Q, b_0\} = H$$

which are formally similar to the ones defining a $N = 2$ algebra. The main difference between (60) and the $N = 2$ algebra is, as it was pointed out in [5], that for the $N = 2$

case the cohomology of b_0 is non trivial and in fact isomorphic to the BRST cohomology of Q, in contrast to what happens for the bosonic string. By a Hodge string we mean one where the b_0 antighost is defined by the G_0 component of the SUSY current G^-:

$$G_0 = Q^* = \oint G^- \tag{61}$$

with Q^* the adjoint of the BRST charge. The name Hodge comes from the fact that the couple (Q, Q^*) satisfies the Hodge relations and in particular the Q, Q^* lemma, i.e. any Q closed form a which is at the same time Q^* exact can be written as QQ^*c for some c.

For the bosonic string theory the couple (Q, b_0) defines an equivariant cohomology [13] by the conditions:

$$\begin{aligned} Q|\chi\rangle &= 0 \\ b_0|\chi\rangle &= 0 \end{aligned} \tag{62}$$

which characterizes the spectrum of physical states and the invariance of string amplitudes with respect to changes of local coordinates. The notion of equivariant cohomology is specially relevant when we work with topological strings. In fact the gravitational descendants appear as pure BRST states $|\chi\rangle = Q|\psi\rangle$ non trivial in the equivariant cohomology: $b_0|\chi\rangle = 0$, $b_0|\psi\rangle \neq 0$ [14, 15]. The existence of these states is on the other hand crucial in the cancel propagator argument computation of contact terms [8, 13, 14].

Let us define the contact term between two chiral fields as follows

$$\int_0^\infty \int_0^{2\pi} d\tau d\phi e^{\tau T_+} e^{\phi T_-} G_{0,+} G_{0,-} \phi_i(1)|\phi_j(0)\rangle \tag{63}$$

where we have used light type coordinates and where $|\phi_j(0)\rangle$ represents the state defined at the boundary of the disc with the field ϕ_j inserted at the origin. A non vanishing contribution to this contact term will come from a Q exact part in the product $\phi_i\phi_j$, non trivial in the equivariant cohomology defined with respect to G_0. We see in this way that gravitational descendants and contact terms are two different aspects of the same fundamental concept, namely the one of equivariant cohomology. If in the definition (63) of contact terms we use for G_0 the super-energy momentum tensor, then, the so defined equivariant cohomology contains all the gravitational descendants and we can symbolically represent the contribution to the contact term as follows

$$\begin{aligned} \phi_i\phi_j &= \sigma_1(\chi) + ... \\ c(\phi_i, \phi_j) &= \chi \end{aligned} \tag{64}$$

These equations make clear the interplay between contact terms and the "matter representation" [14, 15] of gravitational descendants. In order to be precise it is important to make the following remark concerning the contact term (63) for topological matter coupled to topological gravity. The $G's$ appearing in (63), which play in the construction

of string amplitudes as measures on the moduli space, a similar role to the b_0 antighost in the bosonic string, are coming from the integration of the $N = 2$ supermoduli and that is the reason they are defined by the superpartner of the energy momentum tensor. The situation changes dramatically if in equation (63) we use for G_0 the Hodge pair of the BRST charge i.e we use the susy current Q^*. In this case and as a simple consequence of the Q, Q^* lemma, any Q exact state which is Q^* closed is trivial in the equivariant cohomology defined by the Hodge pair (Q, Q^*) and therefore we can not any more interpret the gravitational descendants as non trivial in the (Q, Q^*) equivariant cohomology. In a certain sense and in comparison with the bosonic string, when we work in Hodge strings we loose the richness of gravitational descendants in the equivariant cohomology but we gain a non trivial cohomology for the b_0 antighost. Thinking in these terms it seems natural to expect that part of the physics which is ordinarily associated with the presence of gravitational descendants will appears in Hodge strings as a consequence of the non trivial cohomology for the b_0 antighost. One nice example of this phenomena is the holomorphic anomaly discovered in reference [4, 5] which crucially depends on the non trivial cohomology of the b_0 antighost and on the other hand looks formally very similar to the recursion relations determined by gravitational descendants. This interplay between the holomorphic anomaly and gravitational recursion relations will become clear in the next section in the context of the $t\bar{t}$-contact term algebra.

Why should we work with Hodge strings? In the philosophy underlying this paper Hodge strings and therefore contact terms defined with the G_0 part of the susy current Q^*, seems the natural candidates to represent the variation of topological matter amplitudes under an infinitesimal change of theory, or, in other words, are the natural strings we should use to define the $t\bar{t}$-geometry on the space of 2D-theories. A different way to motivate the concept of Hodge strings is by introducing the notion of covariantization used in ref [5].

In the Hodge case the integral representation (63) of the contact terms already give a good heuristic idea on the connection between the non trivial cohomology of b_0 and the $t\bar{t}$-geometry. In fact if we consider the derivative with respect to $\bar{t}$ of (63), we will get a non trivial contribution from the $\bar{t}$ dependence of G_0 which will be absent in the case the cohomology of G_0 is trivial. This contribution looks formally as the one expected from the $t\bar{t}$-geometry, namely something like

$$\int_0^\infty \int_0^{2\pi} d\tau d\phi e^{\tau T_+} e^{\phi T_-} G_{0,+} Q_{0,+} \bar{\phi}_{\bar{a}}(1)(\phi_i(1)|\phi_j(0)\rangle) = \bar{\phi}_{\bar{a}}(C_{ij}^k|k\rangle) \tag{65}$$

where the derivation of $G_{0,-}$ with respect to $\bar{t}_a$ have been replaced by an antichiral primary field $\bar{\phi}_{\bar{a}}$ which, after contracting with $C_{ij}^k|k\rangle$, will produce the $t\bar{t}$-relation. Using a similar formal argument we can give a functional integral interpretation of the mixed $t\bar{t}$-contact terms we introduce in the previous section. Thus we can interpret $|C(\bar{a}, i)\rangle$ as $\partial_{\bar{a}}|C(\sigma_1, i)\rangle$, where again the derivation with respect to $\bar{a}$ is acting on $G_{0,-}$.

Concerning the dynamical meaning of gravitational descendants in Hodge strings we

can propose the following argument. If we insist in defining the string equivariant cohomology in terms of the Hodge pair (Q, Q^*), as it is the case in the computation of the holomorphic anomaly in [5], then the corresponding physical spectrum would be given by the harmonic zero energy states. If now we want to compute a correlator involving external gravitational descendant states the amplitude we will get will fail to be invariant with respect to the peculiar type of "reparametrizations" generated by G_0[6]. This failure will not affect the invariance with respect to changes of the local coordinates on the world sheet but more likely to "background" $\bar{t}$-independence, the reason being the explicit dependence of G_0 on the $\bar{F}$ part of the lagrangian[7]. This argument makes plausible to associate the $\bar{t}$ dependence of covariant string amplitudes with the contribution of gravitational descendants.

3   The Holomorphic Anomaly

For the case of $\hat{c} = 3$ and for string amplitudes defined by covariant derivatives of the partition function F_g, the $t\bar{t}$-relations imply a $\bar{t}$ dependence of these amplitudes which is known as the holomorphic anomaly. An important ingredient in the derivation of the holomorphic anomaly is the covariant definition of amplitudes in terms of the $t\bar{t}$-connection. This is intimately related to our previous discussion concerning Hogde strings. In fact the covariantization of the amplitudes is forced when we want to interpret them as determining the variation of topological matter amplitudes on the space of couplings. Technically this covariant definition of the string amplitudes is easily done if there exists a vacuum line subbundle $\mathcal{L}$, on the space of couplings, such that the partition function F_g is a section of $\mathcal{L}^{2-2g}$. This can be achieved, in the critical case $\hat{c}=3$, when we reduce the string amplitudes to correlators between truly marginal fields and we identify the space of couplings with the moduli space of the $\hat{c}=3$ $N=2$ super conformal field theory.

In this section we would like to propose a generalization of the holomorphic anomaly to the general case where we do not impose any restriction neither on the value of $\hat{c}$ or on the type of string amplitudes. The logic for this generalization is of course based on our representation of the $t\bar{t}$-geometry in terms of stringy contact terms. It is clear that if we are not considering the critical $\hat{c} = 3$ case we will need to deal with amplitudes involving

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[6]Notice that when we use for G_0 the Hodge pair of the BRST charge Q the condition on physical states to belong to the corresponding equivariant cohomology is stronger than needed in order to push down the string amplitude, as a measure on the augmented moduli space of punctured Riemann surfaces with local coordinates, to the moduli space of punctured Riemann surfaces. This extra condition should be related to background independence.

[7] Using the matter representation of gravitational descendants in Landau-Ginzburg theories (73) with X the Landau-Ginzburg field, we observe that Q will contain the piece $W'dX$ and therefore Q^* a piece like $\bar{W}d\bar{X}$. Thus the failure with respect to the Hodge equivariant condition will depends on the $\bar{t}$-couplings.

relevant fields and/or gravitational descendants.

We will consider first the simplest case of genus zero amplitudes $C_{i_1...i_s}$. In particular, let us begin by the first non-trivial case, namely, the 4-point function $C_{i_1 i_2 i_3 i_4}$ for chiral primary fields. The 3-point function on the sphere is given by

$$C_{i_1 i_2 i_3} = C^n_{i_1 i_2} \eta_{n i_3} \tag{66}$$

where i_1 would corresponds to an operator index, while i_2, i_3 are states indices. Motivated by this, we define

$$C_{i_1 i_2 i_3 i_4} \equiv D_{i_4} C_{i_1 i_2 i_3} = \partial_{i_4} C_{i_1 i_2 i_3} - \Gamma^n_{i_4 i_1} C_{n i_2 i_3} - A^n_{i_4 i_2} C_{i_1 n i_3} - A^n_{i_4 i_3} C_{i_1 i_2 n} \tag{67}$$

with Γ_i and A_i the operator-operator and operator-state connections respectively

$$\Gamma^k_{ij} = A^k_{ij} - A^n_{i0} C^k_{jn} = g_{0\bar{a}} (\partial_i C^{\bar{a}}_{j\bar{b}}) g^{\bar{b}k} \tag{68}$$
$$A^k_{ij} = (\partial_i g_{j\bar{a}}) g^{\bar{a}k}$$

Using that the topological metric η_{ij} is covariantly constant and the integrability of the structure constants (34), we get that definition (67) is symmetric in all indices.

The $\bar{t}$-variation of the 4-point function, $\partial_{\bar{a}} C_{i_1 i_2 i_3 i_4}$, has three type of contributions. From the first term on the r.h.s. of the $t\bar{t}$-equation $\partial_{\bar{a}} A^k_{ij} = [\bar{C}_{\bar{a}}, C_i]^k_j$, we obtain

$$C_{i_1 i_2 n} \bar{C}^{nm}_{\bar{a}} C_{m i_3 i_4} + perm_{(i_1 i_2 i_3 i_4)} \tag{69}$$

which will be generically denoted from now on by "$fact(\bar{C}_{\bar{a}})$". The next contributions come from the $t\bar{t}$-equations for both connections A_i and Γ_i

$$- \bar{C}^n_{\bar{a} i_1} C_{n i_2 m} \eta^{mr} C_{r i_3 i_4} - perm_{(i_1 i_2 i_3 i_4)} \tag{70}$$

It is important here the additional part $A^n_{i0} C^k_{jn}$ of the connection Γ_i, because its $\bar{t}$-derivative provides the term $-\bar{C}^n_{\bar{a} i_4} C_{n i_1 m} \eta^{mr} C_{r i_2 i_3}$, otherwise lacking. Finally, from the additional piece in $\partial_{\bar{a}} \Gamma_i$ still remains

$$\bar{C}^n_{\bar{a} 0} C_{n i_1 m} \eta^{mr} C_{r i_2 s} \eta^{sl} C_{l i_3 i_4} \tag{71}$$

It is useful now to consider Witten's recursion relations [2] for topological strings at genus zero

$$\langle \sigma_{n_1}(i_1) \sigma_{n_2}(i_2) ... \sigma_{n-s}(i_s) \rangle_0 = \tag{72}$$
$$= \sum_{X \cup Y = S} \langle \sigma_{n_1 - 1}(i_1) \prod_{k \in X} \sigma_{n_k}(i_k) \, \alpha \, \rangle_0 \, \eta^{\alpha\beta} \langle \, \beta \prod_{l \in Y} \sigma_{n_l}(i_l) \sigma_{n_{s-1}}(i_{s-1}) \sigma_{n_s}(i_s) \rangle_0$$

where $S = \{i_2 ... i_{s-2}\}$ and α, β are primary fields. For Landau-Ginzburg theories, in which it is possible a matter representation of gravitational descendants, they are given, restricted to the small phase space, by the recursive formula [15]

$$\sigma_n(i) = W' \int^X \sigma_{n-1}(i) + \sum_\alpha \langle \sigma_{n-1}(i) \alpha \rangle_0 \eta^{\alpha\beta} \phi_\beta \tag{73}$$

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with $W(X)$ the superpotential and $W'=\partial_X W$. Let us consider the modified definition of gravitational descendants [14]

$$\tilde{\sigma}_n(i) = W' \int^X \tilde{\sigma}_{n-1}(i) \tag{74}$$

The operators $\tilde{\sigma}_n(i)$ satisfy analogous recursion relations to (72), with the only difference that there are no factorizations with less than three points, namely the subset X should contain at least one point, and all 3-point functions involving a field $\tilde{\sigma}_n(i)$, $n > 0$, vanish. We recall that this last condition was needed in solving the consistency conditions (see paragraph after (55)). In consequence, the gravitational descendants appearing in this paper corresponds to definition (74), instead of (73). We will drop from now on the tilde in $\tilde{\sigma}_n(i)$ to simplify notation.

Using the recursion relations for operators (74)[8], it is easily seen that (70) and (71) are associated to $\sigma_1(i)$ and $\sigma_2(i)$ contributions respectively. Therefore, collecting (69)-(71), we obtain that the $\bar{t}$-variation of the 4-point function can be written in the following way

$$\partial_{\bar{a}} C_{i_1 i_2 i_3 i_4} = fact(\bar{C}_{\bar{a}}) - \sum_{i=1}^{n} \bar{C}_{\bar{a}i}^l C_{\sigma_1(l) i_1..\hat{i}..i_4} + \bar{C}_{\bar{a}0}^l C_{\sigma_2(l) i_1 i_2 i_3 i_4} \tag{75}$$

The existence of a non-vanishing $\bar{t}$-derivative relies in covariantization. Indeed, if we work with non-covariant genus zero amplitudes

$$C_{i_1...i_s} \equiv \partial_{i_s}...\partial_{i_4} C_{i_1 i_2 i_3} \tag{76}$$

the $\bar{t}$-variations are zero due to the holomorphicity of the 3-point function and the commutativity of the t and $\bar{t}$ partial derivatives, $[\partial_i, \partial_{\bar{a}}]=0$.

According to this, we can try to define covariant string amplitudes by requiring that the $\bar{t}$-derivatives are given by the generalization of (75)

$$\partial_{\bar{a}} C_{i_1...i_s} = fact(\bar{C}_{\bar{a}}) - \sum_{i=1}^{s} \bar{C}_{\bar{a}i}^l C_{\sigma_1(l) i_1..\hat{i}..i_s} + \bar{C}_{\bar{a}0}^l C_{\sigma_2(l) i_1...i_s} \tag{77}$$

This expression is symmetric in all the indices by induction because the 3- and 4-point functions are, as it is required.

The holomorphic anomaly equation (77) for generic topological strings can be understood as following from the contact term algebra. In fact, the first contribution to (77) comes from the factorization tensor for the antitopological operators $\bar{a}$ introduced in (57). It is in the second and third terms where resides the main difference with the critical case. The second comes from the mixed contact terms (26), and the third can be interpreted as a bulk contribution, a priori allowed by the symmetric contact term algebra (24) we are working with. Notice that it is licit to give a meaning to covariant derivatives in a

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[8]We are assuming here that Witten's recursion relations are valid for covariantized string amplitudes.

purely stringy way, without making an extra assumption on the existence of a vacuum line subbundle, thanks to the interpretation, worked in section 2.2, of $t\bar{t}$-connection as contact terms.

An important property of the last term in (77) is that, due to charge conservation, it will cancel when we restrict ourselves to a moduli $(t, \bar{t})$-point and marginal perturbations. Indeed, expression (77) reduces to the holomorphic anomaly equation presented in [5] for the critical $\hat{c}=3$ string and moduli perturbations

$$\partial_{\bar{a}} C^g_{i_1...i_s} = fact(\bar{C}_{\bar{a}}) - \sum_{i=1}^s G_{i\bar{a}}(2g-2+s-1)C^g_{i_1..\hat{i}..i_s} \tag{78}$$

where the second contribution in the r.h.s. is given in terms of the Zamolodchikov metric $G_{i\bar{a}}$ [16], an object which requires for its definition the existence of a well defined vacuum line subbundle, something which is not possible in the generic case we are considering. The equivalence of expressions (78) and (77) for a moduli $(t, \bar{t})$-point and marginal perturbations i_j and $\bar{a}$, is achieved because in this particular case the Zamolodchikov metric can be represented as [7]

$$\bar{C}^n_{\bar{a}i} = \frac{g_{i\bar{a}}}{g_{0\bar{0}}}\delta^n_0 = G_{i\bar{a}}\delta^n_0 \ , \tag{79}$$

and the factor $(2g-2+s-1)$, which is the curvature of a Riemann surface of genus g and $s-1$ punctures, corresponds to a dilaton insertion $C_{\sigma_1 i_1..\hat{i}..i_s}$.

It is also important to stress the similarity between the two first contributions to the holomorphic anomaly once we use Witten's recursion relations for gravitational descendants. Again this makes clear the strong interplay between the way the $\bar{a}$ field is factorizing the surface (the first term in (77)), something that in the original derivation of the holomorphic anomaly [5] is due to the non-trivial cohomology of the b_0 antighost in Hodge (covariant) amplitudes, and the factorization rules for gravitational descendants (the second term in (77)).

Based on the previous discussion, it seems plausible to conjecture that the holomorphic anomaly equation obtained is valid for any genus

$$\partial_{\bar{a}} C^g_{i_1...i_s} = fact_g(\bar{C}_{\bar{a}}) - \sum_{i=1}^s \bar{C}^l_{\bar{a}i} C^g_{\sigma_1(l)i_1..\hat{i}..i_s} + \bar{C}^l_{\bar{a}0} C^g_{\sigma_2(l)i_1...i_s} \tag{80}$$

where

$$fact_g(\bar{C}_{\bar{a}}) = \frac{1}{2}\bar{C}^{\alpha\beta}_{\bar{a}} C^{g-1}_{\alpha\beta\, i_1...i_s} + \frac{1}{2}\bar{C}^{\alpha\beta}_{\bar{a}} \sum_{r=0}^g \sum_{X\cup Y=S} C^r_{\alpha\, j_1...j_l} C^{g-r}_{\beta\, j_{l+1}...j_s} \tag{81}$$

with $X = \{j_1...j_l\}$ and $Y = \{j_{l+1}...j_s\}$. As an indication of the validity of (80) we can analyze the genus one case, in which recursion relations analogous to (72) hold [2]. It is straightforward to see that if $C^1_i \equiv \partial_i F^1$ satisfies (80), then $C^1_{ij} \equiv D_i F^1_j = (\partial_i - \Gamma_i)F^1_j$ also does.

Acknowledgments

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